

Distance from a point to a line whose equation is given

Find a formula for the distance, d , from the point $A(x_0, y_0)$ to the line whose equation is given by $y = mx + b$.

Solution

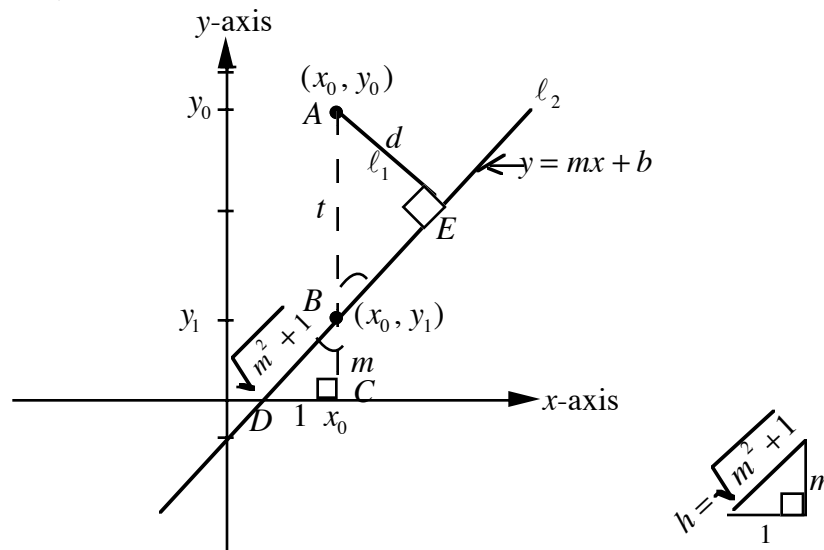
Method 1

Given: The point $A(x_0, y_0)$ and the line $y = mx + b$.

Required: To find a formula for the distance, d , between the point $A(x_0, y_0)$ and the line $y = mx + b$.

Construction: Draw the perpendicular line from $A(x_0, y_0)$ to meet the line $y = mx + b$ (line ℓ_2) at E .

Also, draw $\overline{AC} \perp$ to the x -axis.



Step 1: Let $h = DB$. If the slope of $y = mx + b$ is $m = \frac{m}{1}$, then $BC = m$, $DC = 1$, and $h = \sqrt{m^2 + 1}$.

$\triangle ABE$ and $\triangle DBC$ are similar: (Two angles of $\triangle ABE$ are congruent to two angles of $\triangle DBC$)

$$\therefore \frac{AE}{DC} = \frac{AB}{DB} \quad (\text{Corresponding sides of similar triangles are in proportion})$$

$$\frac{d}{1} = \frac{t}{\sqrt{m^2 + 1}} \quad (1) \quad (AE = d, AB = t, DC = 1, DB = \sqrt{m^2 + 1})$$

Step 2:

$$t = y_0 - y_1$$

$$t = y_0 - (mx_0 + b) \quad (y_1 = mx_0 + b \text{ is obtained by substituting } x_0 \text{ in } y = mx + b)$$

$$t = y_0 - mx_0 - b$$

Step 3: Substitute for t in equation (1)

$$\frac{d}{1} = \frac{y_0 - mx_0 - b}{\sqrt{1 + m^2}}$$

$$d = \frac{y_0 - mx_0 - b}{\sqrt{1 + m^2}}$$

$$\therefore d = \frac{|y_0 - mx_0 - b|}{\sqrt{m^2 + 1}} \quad (\text{We take the absolute value of the numerator in case the numerator is negative})$$

Method 2

Given: The point $A(x_0, y_0)$ and the line $y = mx + b$.

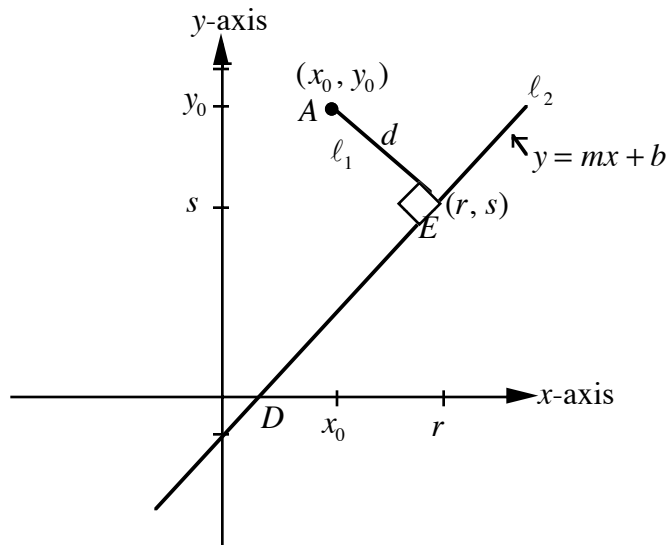
Required: To find a formula for the distance, d , between the point $A(x_0, y_0)$ and the line $y = mx + b$.

Construction: Draw the perpendicular line from the point $A(x_0, y_0)$ to meet the line $y = mx + b$ at E .

Step 1: Let lines ℓ_1 , and ℓ_2 meet at (r, s) . (Figure below)

Let the distance between the points (x_0, y_0) and $(r, s) = d$.

Then $d^2 = (x_0 - r)^2 + (y_0 - s)^2$ or $d = \sqrt{(x_0 - r)^2 + (y_0 - s)^2}$



Step 2: We shall now express d in terms of x_0 , y_0 , m , and b only.

(That is we shall eliminate r , and s from the formula)

$$\text{The slope of line } \ell_1 = \frac{y_0 - s}{x_0 - r}. \quad (1)$$

The slope of line $\ell_2 = m$. (from $y = mx + b$)

$$\text{Therefore slope of } \ell_1 = -\frac{1}{m} \quad (2) \quad (\text{Since } \ell_1 \text{ and } \ell_2 \text{ meet at right angles at } E, \text{ the slopes are negative reciprocals of each other})$$

Equating right-hand-sides of (1) and (2) to each other, we obtain

$$\frac{y_0 - s}{x_0 - r} = -\frac{1}{m}$$

$$my_0 - ms = -x_0 + r \quad (\text{cross-multiplying})$$

$$r = my_0 + x_0 - ms \quad (3) \quad (\text{solving for } r)$$

Since (r, s) is on the line $y = mx + b$

$$s = mr + b \quad (4) \quad (\text{substituting } r \text{ for } x \text{ and } s \text{ for } y.)$$

Step 3: We now solve equations (3) and (4) simultaneously for r and s .

$$\text{Substitute for } s \text{ from (4) in (3): } my_0 + x_0 - m(mr + b) = r$$

$$\text{Now, we solve for } r: my_0 + x_0 - m^2r - mb = r$$

$$my_0 + x_0 - mb = m^2r + r$$

$$my_0 + x_0 - mb = r(m^2 + 1)$$

$$r = \frac{my_0 + x_0 - mb}{m^2 + 1} \quad (5)$$

Substitute for r from (5) in (4). Then we obtain

$$\begin{aligned} s &= m \left[\frac{my_0 + x_0 - mb}{m^2 + 1} \right] + b \\ &= \frac{m^2 y_0 + mx_0 - m^2 b + m^2 b + b}{m^2 + 1} \\ s &= \frac{m^2 y_0 + mx_0 + b}{m^2 + 1} \quad (6) \end{aligned}$$

Step 4: We now substitute right-hand sides of equations (5) and (6) for r and s respectively in the formula $d^2 = (x_0 - r)^2 + (y_0 - s)^2$. Then we obtain

$$\begin{aligned} d^2 &= \left\{ x_0 - \left[\frac{my_0 + x_0 - mb}{m^2 + 1} \right] \right\}^2 + \left\{ y_0 - \left[\frac{m^2 y_0 + mx_0 + b}{m^2 + 1} \right] \right\}^2 \\ &= \frac{\{m^2 x_0 + x_0 - my_0 - x_0 + mb\}^2}{(m^2 + 1)^2} + \frac{\{m^2 y_0 + y_0 - m^2 y_0 - mx_0 - b\}^2}{(m^2 + 1)^2} \\ &= \frac{\{m^2 x_0 - my_0 + mb\}^2}{(m^2 + 1)^2} + \frac{\{y_0 - mx_0 - b\}^2}{(m^2 + 1)^2} \\ &= \frac{\{-m(-mx_0 + y_0 - b)\}^2}{(m^2 + 1)^2} + \frac{\{y_0 - mx_0 - b\}^2}{(m^2 + 1)^2} \\ &= \frac{\{(-m)^2(-mx_0 + y_0 - b)\}^2}{(m^2 + 1)^2} + \frac{\{y_0 - mx_0 - b\}^2}{(m^2 + 1)^2} \\ &= \frac{m^2(y_0 - mx_0 - b)^2}{(m^2 + 1)^2} + \frac{(y_0 - mx_0 - b)^2}{(m^2 + 1)^2} \\ &= \frac{m^2(y_0 - mx_0 - b)^2 + (y_0 - mx_0 - b)^2}{(m^2 + 1)^2} \\ &= \frac{(y_0 - mx_0 - b)^2(m^2 + 1)}{(m^2 + 1)^2} \\ &= \frac{(y_0 - mx_0 - b)^2(m^2 + 1)}{(m^2 + 1)^2} \\ d^2 &= \frac{(y_0 - mx_0 - b)^2}{(m^2 + 1)} \\ d &= \sqrt{\frac{(y_0 - mx_0 - b)^2}{(m^2 + 1)}} \\ &= \frac{\sqrt{(y_0 - mx_0 - b)^2}}{\sqrt{m^2 + 1}} \\ \therefore d &= \frac{|y_0 - mx_0 - b|}{\sqrt{m^2 + 1}} \quad (\text{noting that } \sqrt{a^2} = |a|) \end{aligned}$$

and the derivation is complete.